Statistical properties of the cumulative phase of electromagnetic radiation transmitted through a waveguide with random scatterers

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We calculate the probability distribution function and the average of the cumulative phase of electromagnetic radiation transmitted through the waveguide with randomly positioned dielectric scatterers. The average phase exhibits a crossover from linear to power-law behavior as a function of frequency. A detailed comparison with experimental results is made and a good agreement is found. Our results are consistent with the well-known observation that the scattering mean free path is of the order of the size of scatterers. [S1063-651X(98)09503-8]

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INTRODUCTION

The electromagnetic wave transmitted through random media is fully characterized by two equally important parameters: amplitude and phase. While properties of the wave's amplitude and its intensity have acquired considerable attention over the past decade [1–23], the phase has remained an "unexplored" quantity. Only recently have studies of statistics of the cumulative phase in microwave measurements been reported [22,23]. It has been shown there that the phase is not only central in the underlying wavelike property of the transmitted signal as interference [24], but that it may give direct access to numerous indispensable transport parameters. Among those, to name a few, are the density-of-states, phase, and group velocities. Following the density of phase distribution as a function of sample thickness and frequency, one can also study the transition from the ballistic to the diffusive regime. To advance these investigations further and to fully understand the underlying physics behind the functional behavior of the average phase and phase distribution function found in the experiment, an analytical description of the phase statistics is needed. To our knowledge, no such calculations have been given in the existing literature. The purpose of the present work is to provide analytical analysis of the phase statistics of the electromagnetic radiation transmitted through a waveguide with randomly positioned dielectric scatterers.

The general problem of finding the statistical characteristics of the phase is rather complex theoretically. It is, however, possible to reformulate it in terms of a solvable model. To accomplish this goal, let us look at the transport of an electromagnetic wave with longitudinal component of wave vector k, propagating in a waveguide of length L. A distinctive property of the waveguide of any geometry (rectangular, cylindrical, etc.) is the existence of a discrete set of "transverse" (TE and TM) eigenmodes [25]:

$$\left(\frac{\omega}{c}\right)^2 = k^2 + \left(\frac{\eta n}{R}\right)^2, \tag{1}$$

where ω is the angular frequency of the nth mode, R is the transverse dimension of the waveguide (the width or height in the case of a rectangular waveguide and the radius in the case of a cylindrical tube), c is the speed of the wave inside the empty waveguide, n is a positive integer number, and η is a parameter dependent up on the geometry of the waveguide. In general, there are an infinite number of such eigenmodes in the empty waveguide, even though some of them are suppressed due to symmetry. The incoming electromagnetic radiation (plane-wave or point-source) of wavelength λ can be expanded in the complete set of eigenmodes thus ensuring that all of these modes contribute to the wave transport; λ is the characteristic transport length in this case. Random scatterers, when "placed" in the waveguide, may renormalize some parameters in Eq. (1), while the total number of eigenmodes does not change. However, transport properties of the wave are seriously altered in the presence of random scatterers. It is well known [1–5] that as the wave travels through a random medium its coherent part decays exponentially over a distance of the order of scattering mean free path l due to successive scattering. At the same time the incoherent part builds up gradually and propagates diffusively. Since its phase is completely random and averages out to zero, the total average phase consists only of the superposition of phases of wavelets forming the coherent component. However, the majority of the eigenmodes contributing to the coherent transport are "evanescent," i.e., they exist entirely within a thin layer of the order of several l's from the input face of the waveguide. As the signal propagates deeper inside the waveguide, most of the "coherent" eigenmodes saturate and only a limited number of them exist at the output face of the waveguide. These considerations have recently been tested using numerical simulations [26], where it has been found that only a single coherent mode survives at distances from the input surface much greater than l. An important conclusion appropriate at this point is that under such conditions the waveguide of the length much greater than the scattering mean free path effectively behaves as a quasi-one-dimensional random system.

An additional support of the conclusion made above comes from some specific features of the experimental procedure. In a typical experiment, measurements are performed over the frequency window with width of the order of 20–30 GHz. It follows from Eq. (1) that only a finite number of modes can exist in the restricted frequency interval. Therefore, if a detector is placed at the output face of the waveguide or even outside of the random media, it captures the signal comprised of the finite number of transverse eigenmodes.

In the present work we apply the quasi-one-dimensional approximation for the electromagnetic wave transport in the waveguide with randomly positioned dielectric scatterers to obtain exact analytical results for the total average phase and the density of phase distribution. Our method is to perform a direct mapping of the quasi-one-dimensional (quasi-1D) Maxwell equation onto the one-dimensional Schrödinger equation with a continuous random potential. The Schrödinger equation itself can be converted to the stochastic equation for the spatial derivative of the local phase from which, finally, the general differential equation for the density of the phase distribution function is derived. It is solved explicitly for the total average phase and the density of the phase distribution in the limiting case when both of these parameters are spatially homogeneous. A thorough comparison between analytical results and experimental findings is included.

METHOD OF CALCULATION

Let us start with a cylindrical tube of length L and radius R, filled with randomly positioned dielectric scatterers. The cylinder is a preferred geometry here since it is used in most laboratory setups. Our results are, nevertheless, applicable to a kind of geometry (rectangular, etc.). Scatterers are considered to be pointlike, which leads to the transport mean free path being equal to the scattering mean free path in our case. This is incorrect when the size of scatterers d is of the order of λ. In the present work, however, we are interested in a frequency range for which the condition $\lambda \gg d$ is always satisfied. The plane electromagnetic wave impinges normally at the front surface of the tube and the transmitted signal is detected at its back surface. Following the above-mentioned considerations, we assume that only a finite number of ‘‘transverse’’ eigenmodes is allowed in the tube and, therefore, our problem is reduced to the propagation of a wave in a quasi-1D random system. In order to simplify the calculation even further, we restrict ourselves to the simplest case of a single mode in the waveguide. Under these conditions, the Fourier transform of the amplitude of the electric (magnetic) field $\mathcal{E}(x,t)$ satisfies the wave equation

$$\left\{\frac{\partial^2}{\partial x^2}+k_0^2\epsilon(x)\right\}\mathcal{E}(x,\omega)=0, \tag{2}$$

where $k_0=\omega/c$. The randomness of the system is modeled by the ‘‘dielectric’’ constant $\epsilon(x)=\hat{\epsilon}+\delta\epsilon(x)$, where $\hat{\epsilon}=\epsilon(1-v)$, v is the filling fraction, and ϵ represents the average ‘‘dielectric’’ properties of the medium. The fluctuating part of the ‘‘dielectric constant’’ $\delta\epsilon(x)$ is assumed to be a Gaussian random variable with a zero mean:

$$\langle\delta\epsilon(x)\rangle=0, \quad \langle\delta\epsilon(x)\delta\epsilon(x')\rangle=A\,\delta(x-x'), \tag{3}$$

and the amplitude of the correlations $A\propto\hat{\epsilon}^2 d$, where $\langle\ \rangle$ means averaging over disorder. After employing the following substitution:

$$\frac{\partial\mathcal{E}(x,\omega)}{\partial x}=\mathcal{E}(x,\omega)f(x,\omega),$$

$$\mathcal{E}(x,\omega)=Ce^{\int dx\ f(x,\omega)}=Ce^{\Phi(\omega)}, \tag{4}$$

Eq. (2) reduces to the first-order Ricatti equation for the function $f(x,\omega)$,

$$\frac{df(x,\omega)}{dx}+f^2(x,\omega)=-k_0^2\epsilon(x). \tag{5}$$

From Eqs. (4) we can conclude that $f(x,\omega)$ is nothing but the analog of the spatial derivative of the ‘‘local phase’’ the wave acquires while propagating to a coordinate x in the random medium, whereas $\Phi(\omega)$ can be viewed as a ‘‘total’’ or ‘‘cumulative phase.’’ In order to proceed with further calculations, we note that Eq. (5) is analogous to the 1D Schrödinger equation for a spinless particle of mass m and energy E propagating in the random potential $V(x)$,

$$\frac{df(x,\widetilde{E})}{dx}+f^2(x,\widetilde{E})=2[\widetilde{V}(x)-\widetilde{E}], \tag{6}$$

where the same transformations as given by Eqs. (4) have been performed. In Eq. (6) $\widetilde{E}=4mE$, $\widetilde{V}(x)=4mV(x)$, and Planck's constant is set to unity. An important consequence can be deduced from this analogy. It follows from quasiclassics [27], that the ‘‘number of states’’ $N(E)$, with the energy E and energy density $\rho(E)$, is equal to

$$N(E)=\int dE\ \rho(E)\simeq\frac{1}{2\pi}\int dx\ k(x)=\frac{1}{2\pi}\langle\Phi\rangle_{\widetilde{V}}, \tag{7}$$

where $\langle\ \rangle_V$ means averaging with respect to the random potential. Therefore, the cumulative phase of the classical wave propagating in the 1D random system, averaged with respect to random fluctuations, is directly proportional to the ‘‘number of states’’ in the corresponding quantum-mechanical (QM) problem. Since it is known how to calculate the density of states and the phase distribution for the 1D Schrödinger problem with random potential [28], our next step is to formulate our problem in terms of its QM counterpart. From the direct comparison of Eqs. (5) and (6) we have

$$\widetilde{V}(x)=-\tfrac{1}{2}k_0^2\delta\epsilon(x),$$

$$\widetilde{E}=\tfrac{1}{2}k_0^2\hat{\epsilon}, \tag{8}$$

while the potential-potential correlator takes the following form:

$$\langle\widetilde{V}(x)\widetilde{V}(x')\rangle=\tfrac{1}{4}k_0^4A\,\delta(x-x')=D(k_0)\,\delta(x-x'). \tag{9}$$

An important difference between the propagation of Schrödinger particles and classical waves is that the latter occurs in the ‘‘energy’’-dependent potential, as follows from Eqs. (8). This peculiarity leads to completely different statistical properties of the phase of the classical waves as compared to the

Schrödinger particles. One should note that the states of the electromagnetic field are physically characterized by the wave vector k_0 rather than the ''energy'' E given by Eqs. (8). However, the number of states defined by Eq. (7) is invariant with respect to the change of variables (8). In other words, the number of states as a function of k_0 can be obtained from the result for $N(\tilde{E})$ by means of the simple substitution into Eq. (8) in the form $N(\tilde{E})=N(\tilde{E}(k_0))$. As will be clear below, we only use the invariant quantity $N(\tilde{E})$ in our analysis of the phase distribution. Therefore, we can express the results in terms of $\tilde{E}$ and make use of the QM analogy.

Instead of solving Eq. (6) directly for the ''phase'' $f(x,\tilde{E})$, we use it to derive the equation for the ''density of phase distribution,''

$$P(x,\xi)=\langle \delta(f(x,\tilde{E})-\xi)\rangle_{\tilde{V}}. \tag{10}$$

Therefore, $P(x,\xi)d\xi$ is the probability that $f(x,\tilde{E})$ lies in the interval ξ, $\xi+d\xi$. After taking a partial derivative with respect to the coordinate x from $P(x,\xi)$ and using Furutsu-Novikov's theorem [28],

$$\langle A(x)B(x)\rangle_A=\int dx'\langle A(x)A(x')\rangle\left\langle\frac{\delta B(x')}{\delta A(x')}\right\rangle, \tag{11}$$

we obtain the desired equation for $P(x,\xi)$,

$$\frac{\partial P(x,\xi)}{\partial x}=\frac{\partial}{\partial\xi}\left[\left(\xi^2+2\tilde{E}+2D(k_0)\frac{\partial}{\partial\xi}\right)P(x,k\xi)\right]. \tag{12}$$

This is a Focker-Planck equation with a ''diffusion coefficient'' $D(k_0)$.

RESULTS AND DISCUSSION

Equation (12) can only be solved numerically for $P(x,\xi)$. In a large system, however, where the number of states is an extensive parameter, the density of phase distribution should not depend on the spatial coordinate x. In the case of the spatially homogeneous $P(\xi)$, Eq. (12) reduces to

$$\left(\xi^2+2\tilde{E}+2D(k_0)\frac{\partial}{\partial\xi}\right)P(\xi)=C_0(D,\tilde{E}). \tag{13}$$

Here $C_0(D,\tilde{E})$ is a constant. It can be shown [28] that $C_0(D,E_0)=N(E_0)/(DL)$, where $N(E_0)$ is the number of states with the energy $\tilde{E}$ less than E_0. From Eq. (7), it follows that the total phase is related to the constant on the right-hand side of Eq. (13) as

$$\langle\Phi\rangle_{\tilde{V}}=2\pi D(k_0)C_0(D,E_0)L. \tag{14}$$

Equation (13) should be supplemented by the renormalization condition,

$$\int_{-\infty}^{+\infty}d\xi\, P(\xi)=1, \tag{15}$$

which provides an additional definition of $C_0(D,\tilde{E})$. Solving Eq. (13) together with Eq. (15) we obtain

$$P(\xi)=\frac{C_0}{2}\exp\left[-\frac{1}{D(k_0)}\left(\frac{\xi^3}{6}+\tilde{E}\xi\right)\right]\times\int_{-\infty}^{\xi}du\,\exp\left[\frac{1}{D(k_0)}\left(\frac{u^3}{6}+\tilde{E}u\right)\right]. \tag{16}$$

Making use of Eqs. (14) and (15), we finally arrive at the following expression for $\langle\Phi(\tilde{E})\rangle$:

$$\langle\Phi(\tilde{E})\rangle=\frac{\sqrt{8\pi D(k_0)}L}{\int_0^{+\infty}\frac{du}{\sqrt{u}}\exp\left[-\frac{1}{D(k_0)}\left(\frac{u^3}{24}+\tilde{E}u\right)\right]}. \tag{17}$$

We would like to mention that the integral in the denominator of Eq. (17) can be calculated exactly, with the result of integration expressed in terms of hypergeometric functions. The form of Eq. (17) is, however, more convenient for both the analytical estimation of limiting cases and numerical calculations. Two important limiting cases, long-wavelength and short-wavelength, can be distinguished. In the case of long wavelengths (low frequencies), $\tilde{E}\to 0$, $k_0\to 0$, where the wavelength is much larger than the scale of inhomogeneities, disorder should not have a significant impact on the cumulative phase, since waves hardly get scattered before leaving the waveguide. We obtain the linear growth of the ''average cumulative phase'' with a frequency ν,

$$\langle\Phi(\nu)\rangle\simeq\sqrt{2\hat{\epsilon}}k_0L\propto\nu. \tag{18}$$

The numerical coefficient in Eq. (16) is indeed independent of disorder. It can also be seen from Eq. (15) that in the case of low frequencies the typical path length of the wave inside the waveguide is of the order of L. In the opposite case of short wavelengths ($\tilde{E}\to\infty$, $k_0\to\infty$), the ''average phase'' exhibits much steeper growth, namely $\nu^{4/3}$, as a function of frequency,

$$\langle\Phi(\nu)\rangle\simeq\sqrt{\pi}C\left(\frac{A}{4}\right)^{1/3}k_0^{4/3}L\propto\nu^{4/3}, \tag{19}$$

where $C=\int_0^\infty du\,\exp[-u^6/3]$ is a constant. It can be attributed to the fact that waves with a short wavelength experience a lot of scattering events inside the waveguide. As a result, these waves stay longer in the waveguide and travel average distances much greater than the length of the waveguide. Using Eqs. (14) and (15), the homogeneous density of phase distribution $P(\xi)$ can be presented in the form

$$P(\xi)=\frac{N(\tilde{E})}{2D(k_0)}\exp\left[-\frac{1}{D(k_0)}\left(\frac{\xi^3}{6}+\tilde{E}\xi\right)\right]\times\int_{-\infty}^{\xi}du\,\exp\left[\frac{1}{D(k_0)}\left(\frac{u^3}{6}+\tilde{E}u\right)\right]. \tag{20}$$

We plot $P(\xi)$ in Fig. 1 for three different frequencies of 1 Ghz [case (a)], 15 Ghz [case (b)], and 50 Ghz [case (c)]. It is clear from the figure, that the probability of acquiring a large phase inside the medium is much greater at higher frequencies.

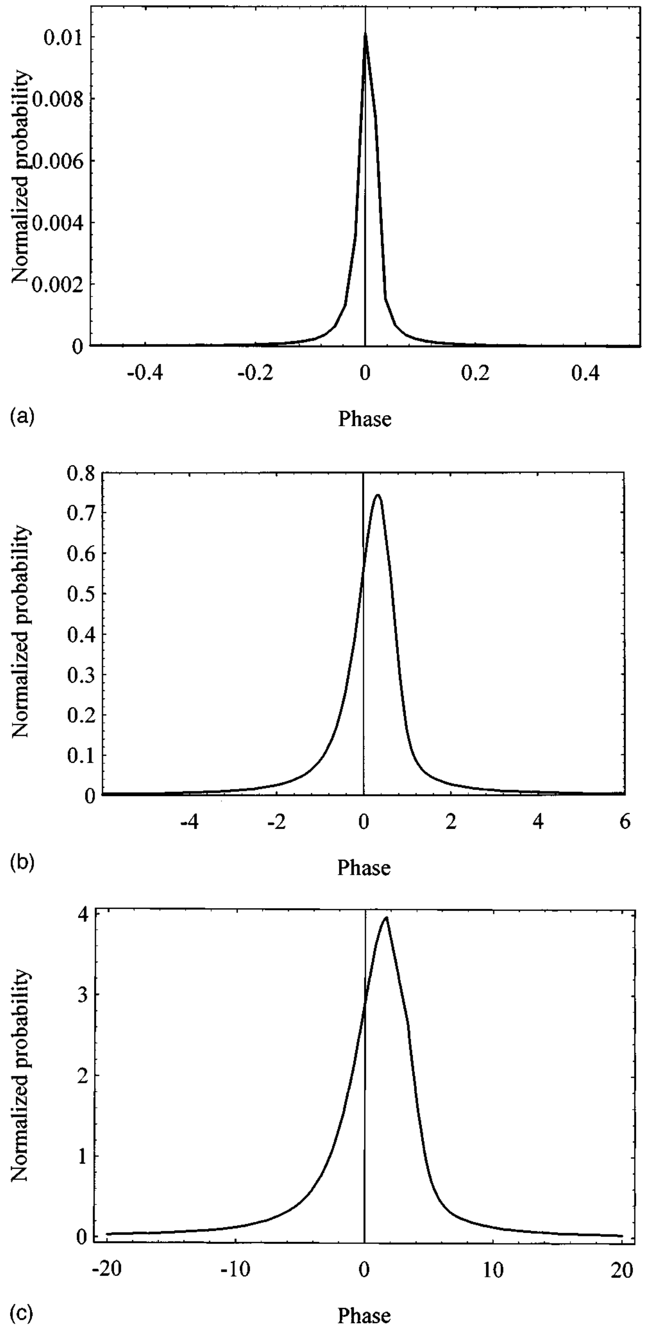

FIG. 1. The ''normalized'' density of the phase distribution function $\widetilde{P}(\Phi)=P(\Phi)/[P_0/2D(k_0)]$ as a function of Φ for three different frequencies (a) $\nu_1=1$ Ghz, (b) $\nu_2=15$ Ghz, and (c) $\nu_3=50$ Ghz.

In order to understand how relevant our results are to the measurements performed in waveguides with random scatterers, we will make a comparison to the experimental data reported in Ref. [23]. In this work measurements of the phase of microwave radiation transmitted through a sample of randomly positioned $\frac{1}{2}$-in. polystyrene spheres ($\epsilon\approx 3$) at a volume fraction of 0.52 were performed. The sample of length $L=110$ cm was contained within a 7.6-cm-diam copper tube and the frequency was swept from 3 to 26 GHz. From Eq. (1) it is possible to estimate the number of modes existing in this frequency window. The frequency span $\Delta\nu$ between nth and $(n+1)$th modes can be estimated as

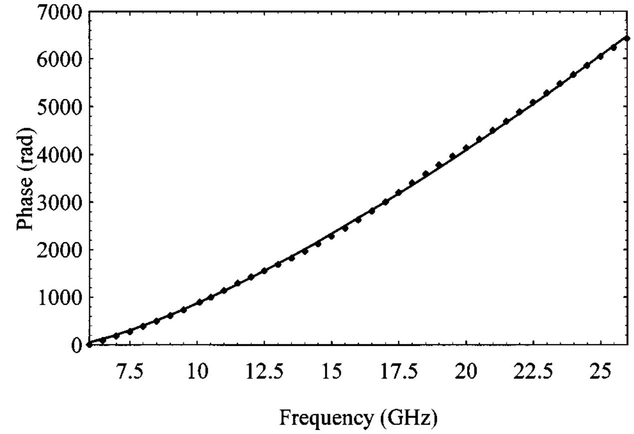

FIG. 2. The average cumulative phase as a function of frequency given by Eq. (1) (solid line) and from Ref. [23] (dotted line) for the value of $A=3.332$.

$$\Delta\nu\approx\frac{\eta c}{2\pi R\sqrt{\hat{\epsilon}}}, \tag{21}$$

where η is expressed in terms of roots of the Bessel function $J_0(x)$ ($\eta\approx 2\pi$). Substituting parameters of Ref. [23] into Eq. (19) we find that $\Delta\nu\approx 6.58$ Ghz, which corresponds to approximately three modes. This means that experimental data obtained in Ref. [23] can be very well described by the quasi-1D approximation. Unfortunately, it is impossible to make a comparison between our result for the density of phase distribution given by Eq. (18) and the experimental data reported in Ref. [23]. The reason is that the probability distribution in the paper by Sebbah *et al.* is presented as a function of the fluctuation of the phase from its ensemble average value $\delta\Phi=\Phi-\langle\Phi\rangle$, whereas our $P(\Phi)$ has the total phase Φ as its argument. In addition, $P(\Phi)$ of Ref. [23] is computed using the cumulative phase measured at each frequency for every configuration and then averaged over the whole frequency interval. In our case, $P(\Phi)$ is calculated for a single frequency. We can, however, compare our analytical expression for the total average phase with experimental data. The right way to do that is to substitute values of L, c, and ϵ into Eq. (15) and then plot $\langle\Phi\rangle$ as a function of frequency. The problem is that the value of the amplitude of the correlations A is not known. Thus, we have to perform a fit of Eq. (15) to experimental data, results of which are shown in Fig. 2. Excellent agreement between the theory and experiment, as seen in Fig. 2, is achieved for the value of $A=3.332$. The amplitude of the correlations A is quite an important parameter, since it is an integral part of the scattering mean free path. It can also be used as an independent test of our results. The expression for l in terms of A in the 1D case can be found by applying the standard diagrammatic approach (see, for example, Refs. [1–4]), which gives the result $l=4/Ak_0^2$. Substituting the value of A found from the fit into this formula, we obtain the value of $l=1.23$ cm at the frequency $\nu=10$ GHz. It is a reasonable figure, since it perfectly supports the well-known result [1,2] that the scattering mean free path is of the order of the size of scatterers. In the case of Ref. [23] scatterers had the diameter equal to 1.27 cm. These estimations provide an additional strong support of our initial assumption that the propagation of the electromagnetic wave in a waveguide with randomly distributed dielectric scatterers can be simulated by the transport in the quasi-1D random system.

CONCLUSION

In summary, we have calculated the probability distribution and the average of the cumulative phase of the electromagnetic wave transmitted through the waveguide with randomly positioned dielectric scatterers. Our calculations are based on the approximation of the waveguide with randomly distributed dielectric scatterers by the quasi-1D random system. We have obtained the result that the average phase exhibits a crossover from linear to power-law behavior as a function of frequency. Excellent agreement is found between analytical and experimental results, thus supporting our approximation.

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